Oxford Cambridge and RSA

Thursday 13 June 2019 – Afternoon

AS Level Further Mathematics B (MEI)

Y415/01 Mechanics b

Time allowed: 1 hour 15 minutes

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **8** pages.

1 A small object of mass 5 kg is attached to one end of each of two identical parallel light elastic strings. The upper ends of both strings are attached to a horizontal ceiling.
The object hangs in equilibrium at R, with the extension of each string being 0.1 m, as shown in Fig. 1.

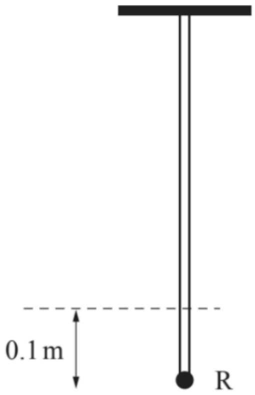


Fig. 1

(a) Find the stiffness of each string. **[3]**

One of the strings is now removed and the object initially falls downwards. The object does not return to R at any point in the subsequent motion.

(b) Suggest a reason why the object does not return to R. **[1]**

a)

$T = kx$

stiffness

for equilibrium: $\underbrace{2k \times 0.1}_{\text{total tension}} = 5g$

$\rightarrow 2k = 50g$

$k = 25g \text{ Nm}^{-1}$ writing units is a good habit to get into

$= 245 \text{ Nm}^{-1}$

b) the string may have stretched beyond its elastic limit ('new' unextended length), or the mass may have lost energy in a non-elastic collision

2 A particle P of mass m travels in a straight line on a smooth horizontal surface.
At time t, P is a distance x from a fixed point O and is moving with speed v away from O. A horizontal force of magnitude $3mt$ acts on P, in a direction away from O.

(a) Show that $\frac{d^2x}{dt^2} = 3t$. **[1]**

(b) Verify that the general solution of this differential equation is $x = \frac{1}{2}t^3 + At + k$, where A and k are constants. **[2]**

(c) Given that $x = 6$ and $v = 12$ when $t = 1$, find the values of A and k. **[4]**

a) Newton's 2nd law for constant mass: $F = ma$

so $3mt = ma = m\frac{d^2x}{dt^2} \Rightarrow \frac{d^2x}{dt^2} = 3t$

b) differentiate given solution: $\dot{x} = \frac{3}{2}t^2 + A$

$\ddot{x} = 3t \Rightarrow$ consistent

c) $x=6$, $v=12$, $t=1$: $6 = \frac{1}{2}(1) + A(1) + k \leftarrow x$

$12 = \frac{3}{2}(1) + A \leftarrow v$

$\therefore \frac{21}{2} = A$

$k = 6 - \frac{1}{2} - \frac{21}{2}$

$k = -5$

3 A particle Q of mass m moves in a horizontal plane under the action of a single force $\mathbf{F}$.

At time t, Q has velocity $\begin{pmatrix} 2 \\ 3t-2 \end{pmatrix}$.

(a) Find an expression for $\mathbf{F}$ in terms of m. **[2]**

At time t, the displacement of Q is given by $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$. When $t = 1$, Q is at the point with position vector $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$.

(b) Find the equation of the path of Q, giving your answer in the form $y = ax^2 + bx + c$, where a, b and c are constants to be determined. **[7]**

(c) What can you deduce about the path of Q from the value of the constant c you found in part **(b)**? **[1]**

a) $\underline{v} = \begin{pmatrix} 2 \\ 3t-2 \end{pmatrix} \Rightarrow \underline{a} = \frac{d\underline{v}}{dt} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \Rightarrow \underline{F} = m\underline{a} = m\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

b) $\underline{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \int \underline{v}\, dt \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t + c \\ \frac{3}{2}t^2 - 2t + d \end{pmatrix}$ different constants

$t = 1: \underline{r} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \Rightarrow 2(1) + c = 4$ so $c = 2$

$\frac{3}{2}(1) - 2(1) + d = -4$ so $d = -\frac{7}{5}$

$x = 2t + 2$ & $y = \frac{3}{2}t^2 - 2t - \frac{7}{5}$

rearrange to eliminate t:

$t = \frac{1}{2}x - 1$

sub in expression for y: $y = \frac{3}{2}\left(\frac{1}{2}x - 1\right)^2 - 2\left(\frac{1}{2}x - 1\right) - \frac{7}{2}$

$= \frac{3}{2}\left(\frac{1}{4}x^2 - x + 1\right) - x + 2 - \frac{7}{2}$

$= \frac{3}{8}x^2 - \frac{3}{2}x + \frac{3}{2} - x - \frac{3}{2}$

$y = \frac{3}{8}x^2 - \frac{5}{2}x$

c) $C=0$ so particle passes through the origin

(NOT starts @ origin)

4 Two uniform discs, A of mass 0.2 kg and B of mass 0.5 kg, collide with smooth contact while moving on a smooth horizontal surface.
Immediately before the collision, A is moving with speed $0.5\,\text{m s}^{-1}$ at an angle α with the line of centres, where $\sin\alpha = 0.6$, and B is moving with speed $0.3\,\text{m s}^{-1}$ at right angles to the line of centres. A straight smooth vertical wall is situated to the right of B, perpendicular to the line of centres, as shown in Fig. 4. The coefficient of restitution between A and B is 0.75.

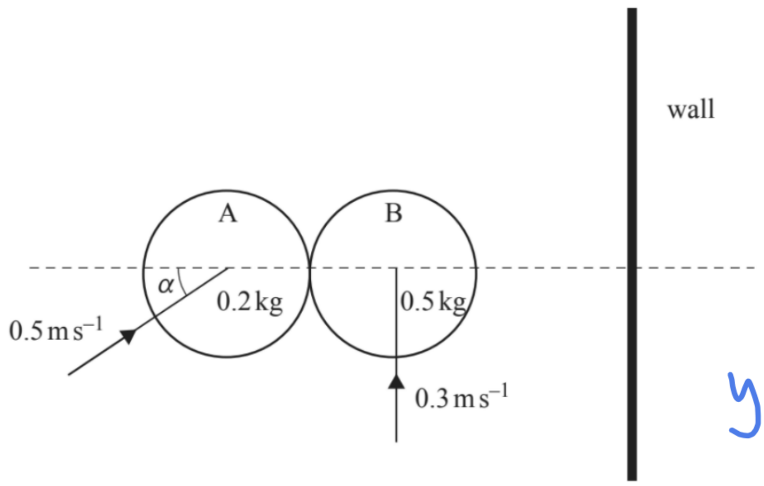


Fig. 4

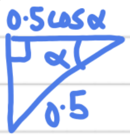
(a) Find the speeds of A and B immediately after the collision. **[8]**

(b) Explain why there could be a second collision between A and B if B rebounds from the wall with sufficient speed. **[1]**

(c) Find the range of values of the coefficient of restitution between B and the wall for which there will be a second collision between A and B. **[3]**

(d) How does your answer to part **(b)** change if the contact between B and the wall is not smooth? **[1]**

a) conservation of momentum in x direction:



$$0.2 \times 0.5\cos\alpha + 0 = 0.2v_a + 0.5v_b$$

coefficient of restitution: $-e = \dfrac{v_b - v_a}{u_b - u_a}$

so $v_b - v_a = -0.75(0 - 0.5\cos\alpha)$

$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-0.6^2} = 0.8 \Rightarrow v_b - v_a = 0.3$ ①

$0.2v_a + 0.5v_b = 0.08$ ②

solve simultaneous equations:

$5 \times ②: v_a + 2.5v_b = 0.4$

$① + 5 \times ②: 3.5v_b = 0.7$

$v_b = 0.2$

sub in: $0.2 - v_a = 0.3$ parallel to x

$v_a = -0.1$

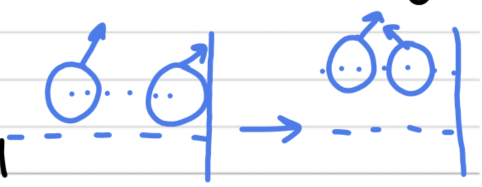
momentum only transferred in x-direction, so after collision, velocities in y direction are:

$v_A = 0.5\sin\alpha = 0.3$, $v_B = 0.3$

$\therefore$ speed of A $= \sqrt{0.3^2 + 0.1^2} = \sqrt{0.1} \approx 0.316\ ms^{-1}$

" " $B = \sqrt{0.3^2 + 0.2^2} = \sqrt{0.13} \approx 0.361\ ms^{-1}$

b) they could collide again $\because$ the velocity perpendicular to the line of centres is 0.3 for both A & B, so they will remain in line over time



c) to 'catch up' with A, after hitting wall B's speed must $> 0.1 ms^{-1}$ towards A

$e = \frac{v_B}{u_B}$ } before & after colliding with wall $\Rightarrow v_B > 0.1$ so $e > \frac{0.1}{0.2}$ $\therefore e > 0.5$

d) B's speed along the wall wouldn't be conserved, so wouldn't be the same as A's, $\therefore$ no 2nd collision

5 Fig. 5 shows the curve with equation $y = -x^2 + 4x + 2$.
The curve intersects the x-axis at P and Q. The region bounded by the curve, the x-axis, the y-axis and the line $x = 4$ is occupied by a uniform lamina L. The horizontal base of L is OA, where A is the point (4, 0).

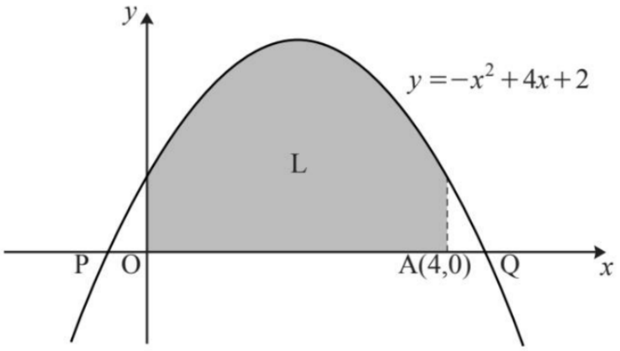


Fig. 5

(a) **(i)** Explain why the centre of mass of L lies on the line $x = 2$. **[1]**

(ii) In this question you must show detailed reasoning.

Find the y-coordinate of the centre of mass of L. **[7]**

(b) L is freely suspended from A. Find the angle AO makes with the vertical. **[2]**

The region bounded by the curve and the x-axis is now occupied by a uniform lamina M. The horizontal base of M is PQ.

(c) Explain how the position of the centre of mass of M differs from the position of the centre of mass of L. **[2]**

a) i. the lamina's curve is symmetrical about $x=2$, so L is, too

ii. $\bar{y} = \dfrac{\int_a^b \frac{1}{2}y^2\,dx}{\int_a^b y\,dx}$, $M\bar{y} = \int_a^b \frac{1}{2}\rho y^2\,dx$, $M = \int_a^b \rho y\,dx$

area $= \int_0^4 (-x^2+4x+2)\,dx = \left[-\frac{1}{3}x^3 + 2x^2 + 2x\right]_0^4$

$= \frac{56}{3}$

$\frac{1}{2}y^2$ integral: $\frac{1}{2}\int_0^4 (-x^2+4x+2)^2\,dx = \frac{1}{2}\int_0^4 (x^4 - 8x^3 + 12x^2 + 16x + 4)\,dx$

$$= \frac{1}{2}\left[\frac{1}{5}x^5 - 2x^4 + 4x^3 + 8x^2 + 4x\right]_0^4$$

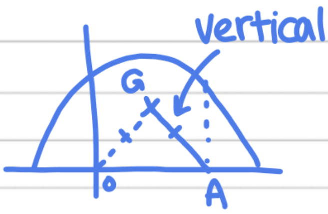
$$= \frac{232}{5}$$

$$\Rightarrow \bar{y} = \frac{232}{5} \div \frac{56}{3}$$

$$= \frac{87}{35}$$

$$= 2.49$$

b) freely suspended $\Rightarrow$ C.O.M. directly below point of suspension



C.O.M. $(2, 2.49)$

symmetric about $x=2$ $\therefore$ $\angle OAG = \angle AOG$ (isosceles)

$\tan OAG = \frac{2.49}{2}$ $\therefore$ $\angle OAG$ = angle to vertical $= 51.2°$

c) $\bar{x}$ remains the same (same symmetry)

$\bar{y}$ will decrease $\because$ part of M not in L (added mass) is closer to the x-axis, adding weight below 2.49

6 A smooth solid hemisphere of radius a is fixed with its plane face in contact with a horizontal surface.

The highest point on the hemisphere is H, and the centre of its base is O. A particle of mass m is held at a point S on the surface of the hemisphere such that angle HOS is 30°, as shown in Fig. 6.

The particle is projected from S with speed $0.8\sqrt{ag}$ along the surface of the hemisphere towards H.

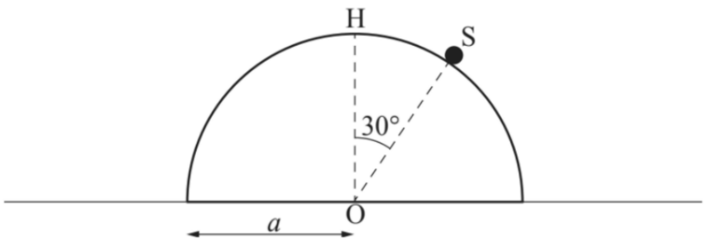


Fig. 6

(a) Show that the particle passes through H without leaving the surface of the hemisphere. **[4]**

After passing through H, the particle passes through a point Q on the surface of the hemisphere, where angle HOQ = θ°.

(b) State, in terms of g and θ, the tangential component of the acceleration of the particle when it is at Q. **[1]**

The particle loses contact with the hemisphere at Q and subsequently lands on the horizontal surface at a point L.

(c) Find the value of $\cos\theta$ correct to 3 significant figures. **[4]**

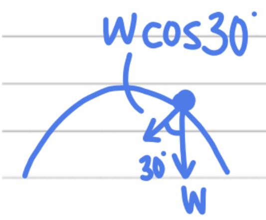
(d) Show that OL = ka, where k is to be found correct to 3 significant figures. **[5]**

END OF QUESTION PAPER

a) to remain on hemisphere, $R > 0$. particle must have enough energy to reach H.

centripetal

@ S, F towards centre = $\dfrac{mv^2}{r} = \dfrac{m(0.8\sqrt{ag})^2}{a} = 0.64mg$



W towards centre = $mg\cos 30°$

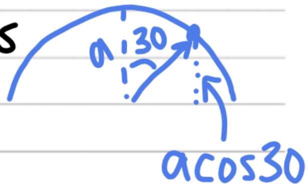
$= 0.866mg$

$R = W_{SO} - F_{SO}$

$0.866mg > 0.64mg \Rightarrow R > 0$, P stays on hemisphere

Energy @ S = K.E. + G.P.E. $= \frac{1}{2}m(0.64ag) + mga\cos 30°$

E @ S is > G.P.E @ H, mga, so P reaches H & passes through H



b) $g\sin\theta$ 'state' implies you shouldn't need to do any calculations

c) loses contact $\Rightarrow R = 0$

@ Q, $mg\cos\theta = \frac{mv^2}{a}$ ①

E conserved: $\frac{1}{2}m(0.64ag) + mga\cos 30° = \frac{1}{2}mv^2 + mga\cos\theta$

sub in ①: $0.32amg + mga\cos 30° = \frac{amg\cos\theta}{2} + mga\cos\theta$

$\Rightarrow \cos\theta = \frac{1}{3}(0.64 + 2\cos 30°)$

keep in exact form for as long as possible

d) $v^2 = 7.748664a$ using $v^2 = ag\cos\theta$

vertical motion (SUVAT): $a\cos\theta = 2.7836\sqrt{a}\sin\theta t + \frac{1}{2}gt^2$

from quadratic formula & $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-0.791^2} = 0.611$

$t = 0.2638\sqrt{a}$

$OL = OQ + v\cos\theta t = a\sin\theta + 2.7836\sqrt{a}\cos\theta(0.2638\sqrt{a})$

$= 1.19a$